

# ALTERNATING PATHS OF FULLY PACKED LOOPS AND INVERSION NUMBER

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**ABSTRACT.** We consider the set of alternating paths on a fixed fully packed loop of size  $n$ ,  $\phi_0$ . This set is in bijection with the set of fully packed loops of size  $n$ . Furthermore, for a special choice of  $\phi_0$ , we demonstrate that the set of alternating paths are nested osculating loops, which we call Dyck islands. This set is also shown to be a self-dual distributive lattice. In the hope of shedding light on a bijective proof of the alternating sign matrix-descending plane partition (ex-)conjecture, we provide a simple characterization of the inversion number in terms of loops of Dyck islands.

## 1. INTRODUCTION

The motivation for studying alternating paths of fully packed loops began with the online note of Ayer and Zeilberger [2] on an attempt to prove the Razumov-Stroganov conjecture. This note introduced the notion of an alternating path of a fully packed loop and described their action on the underlying link patterns in simple example cases. Furthermore, they conjectured the existence of an algorithm for finding an alternating path that would implement the pullback of the local XXZ Hamiltonians into the space of fully packed loops in such a way that would provide a solution to the Razumov-Stroganov conjecture. The RS conjecture has since been solved by Cantini and Sportiello's detailed analysis [5] of Wieland's gyration operation [10] on fully packed loops, but the question of the existence of an algorithm with the desired properties remains open.

On another note, Striker [8] and Behrend and Knight [4] independently studied the notion of the alternating sign matrix polytope. In particular, Striker gives a nice characterization of the face lattice of this polytope in terms of what she calls doubly directed regions of flow diagrams [8]. Recast into the fully packed loop picture, this is described as follows: Given any two fully packed loops, there is an alternating path (possibly a disjoint union of alternating loops) between them. Then given some collection of fully packed loops of size  $n$ , one can consider the union of all alternating paths between pairs of fully packed loops. This union represents the smallest face of the alternating sign matrix polytope which contains all of the fully packed loops in the collection.

In what follows, this paper is roughly divided into two segments. In the first, we wish to use alternating paths to construct a new height function for fully packed loop configurations. This in turn will give a new lattice structure on the set of fully packed loops with the nice property that it is self-dual. The construct is reminiscent of the one considered by Propp [7] where he considers lattice structures constructed from arbitrary directed graphs. It would be of great interest to understand how this lattice is related to the self-dual lattice of descending plane partitions as described by Mills, Robbins and Rumsey [6] in 1983. The second half of

this paper then takes a step towards answering this question by first establishing the connection between the shape of the height function and the inversion number of an alternating sign matrix. In particular, we show that the inversion number of an alternating sign matrix can be decomposed as follows:

$$\text{inv}(A) = \sum_{i=1}^{\ell} \text{inv}(\gamma_i) - k$$

where  $\gamma_1, \dots, \gamma_{\ell}$  are alternating loops, the union of which correspond to the alternating sign matrix  $A$ , and  $k$  is the number of off-diagonal osculations of these loops. The quantity  $\text{inv}(\gamma_i)$  will be shown to be dependent only on the diameter of the loop and the number of osculations on the diagonal.

In the same paper [6], Mills, Robbins and Rumsey made the refined conjecture that the sets of alternating sign matrices of size  $n$  with position of 1 in the first row equal to  $k$ , inversion number  $m$ , and  $\ell$  entries equal to  $-1$  are in bijection with the descending plane partitions with all entries less than or equal to  $n$ , with  $m$  parts,  $\ell$  special parts, and  $k$  parts equal to  $n$ . A recent paper of Behrend, Di Francesco, and Zinn-Justin [3] establishes this long outstanding conjecture, but the search for a bijective proof is still open. To date, bijections [1, 9] exist between the set of permutation matrices of size  $n$  and descending plane partitions with all parts less than or equal to  $n$ , with no special parts. Hopefully, the ideas presented here will give clues on how to extend the bijection to all alternating sign matrices.

## 2. TERMINOLOGY

Let us now clarify the terminology which will be used throughout the paper.

**Definition 2.1.** A *fully packed loop* of size  $n$  is a connected graph arranged in an  $n \times n$  grid such that there are  $n^2$  internal vertices of degree 4, and  $4n$  external vertices of degree 1. Edges of the graph are colored either light or dark such that all internal vertices are incident to two light edges and two dark edges—this is the six-vertex condition. Furthermore, edges incident to the vertices of degree 1 alternate in color in the manner seen in Figure 2.

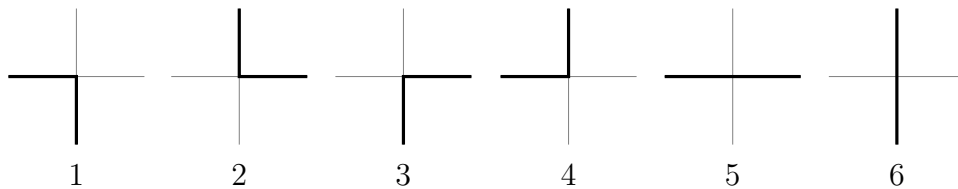
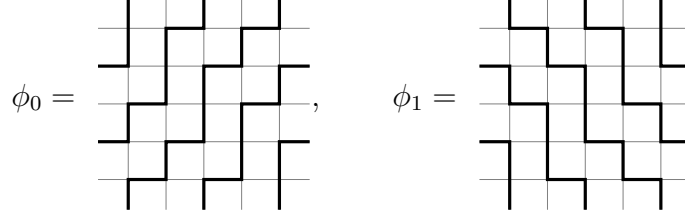


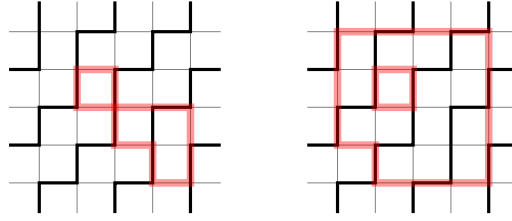
FIGURE 1. The six-vertex condition

It is well known that there exists a bijection between fully packed loops of size  $n$  and alternating sign matrices of size  $n$ . Vertices of type 1 – 4 correspond to 0, and vertices of type 5 and 6 correspond to 1 and  $-1$  subject to the alternating sign condition.

**Definition 2.2.**  $\phi_0$  is the fully packed loop associated to the identity matrix. Likewise,  $\phi_1$  is the fully packed loop associated to the skew-identity matrix.

FIGURE 2. The fully packed loops  $\phi_0$  and  $\phi_1$ .

**Definition 2.3.** An *alternating path* is a collection of edges of a fully packed loop which form lattice path loops and for which the edge color alternates at each step along the loops. For specificity, an *alternating loop* is a single loop which has alternating edge colors, and in contrast, an alternating path is a union of such loops, and a *plaquette flip* is an alternating loop surrounding a  $1 \times 1$  box.

FIGURE 3. Alternating paths on  $\phi_0$ .

**Definition 2.4.** A *Dyck island* of size  $n + 1$  is an  $n \times n$  Young tableau filled with entries,  $\delta_{ij}$  for  $1 \leq i, j \leq n$ , from  $\{0, 1, 2, 3, \dots\}$  according to the following rules:

- $\delta_{ij} \geq \delta_{i',j'}$  whenever  $i \geq j, i \leq i',$  and  $j \geq j'$
- $\delta_{ij} \geq \delta_{i',j'}$  whenever  $i \leq j, i \geq i'$  and  $j \leq j'$
- $\delta_{ij} = 0$  or  $1$  if  $i, j \in \{0, n\}$
- $|\delta_{ij} - \delta_{(i+1)j}| \leq 1$  and  $|\delta_{ij} - \delta_{i(j+1)}| \leq 1$ .

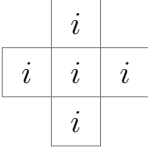
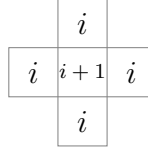
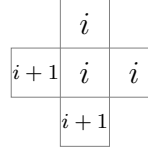
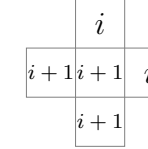
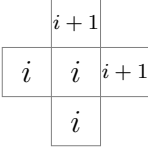
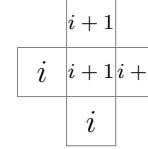
In words, the above definition tells us the following: Fix a box on the diagonal. Entries of boxes above and to the right are weakly decreasing, by increments of at most 1 per step. Likewise, entries of boxes below and to the left are weakly decreasing, by increments of at most 1 per step. Furthermore, boxes along the boundary can only take the values 0 or 1. As we shall soon see, superimposing a Dyck island over the fully packed loop  $\phi_0$  specifies an alternating path along the boundaries of level sets. Thus Dyck islands are a height function representation.

0	0	0	0
0	1	0	0
0	0	1	1
0	0	0	1

1	1	1	1
1	2	1	1
1	1	1	1
0	1	1	1

FIGURE 4. Two example Dyck islands.

It is possible to construct all Dyck islands inductively according to the following local update rules:

-   $\leftrightarrow$   when the entry to be updated is on the diagonal.
-   $\leftrightarrow$   when the entry to be updated is above the diagonal.
-   $\leftrightarrow$   when the entry to be updated is below the diagonal.

For the purpose of making sense of the update rules along the boundary, assume that the Dyck island is padded along the boundary by 0 entries. Let us call the diagrams on the left hand side of the update rules *inside corners* and the diagrams on the right hand side *outside corners*. So the local update rules swap inside corners with outside corners and vice versa.

It follows from the above local rules that the boundaries of level sets are loops which can be nicely characterized as two Dyck paths with one forming the northeast boundary and the other forming the southwest boundary. When it is convenient, we shall abuse notation by using the height function picture and boundaries of level sets picture interchangeably.

### 3. THE ALTERNATING PATHS OF $\phi_0$

**Proposition 3.1.** *Fix a fully packed loop  $\phi$  of size  $n$ . There is an alternating path between  $\phi$  and any other fully packed loop  $\phi'$ . Thus, the set of alternating paths on  $\phi$  is in bijection with the set of fully packed loops of size  $n$ .*

*Proof.* It is clear that given an alternating path of the given fully packed loop  $\phi$ , applying a color flip to the alternating path yields a new fully packed loop. In this way, for each alternating path, there is at most one associated fully packed loop. In order to show that for each alternating path there is at least one fully packed loop, we show how to construct an alternating path given two fully packed loops.

Let two different  $\phi$  and  $\phi'$  be given. In general as a consequence of the six vertex condition, we have that each vertex may differ in the coloring of 0, 2, or 4 edges.  $\phi$  and  $\phi'$  must differ by at least 1 edge, which we give the label  $e$ . Thus,  $\phi$  and  $\phi'$  must also differ at another edge incident to the same vertex, say  $e'$ . Furthermore, the edges  $e$  and  $e'$  must have different colors. Thus at each vertex incident to an alternating path, the path has degree two or four and the edges of such paths have colors which alternate. At vertices of degree 4 of type 1 – 4, there is no ambiguity with regards to how the four edges should be cut up into two alternating path segments. At vertices of type 5 and 6, we give the following interpretation: Let the north and east edges be connected into a path, and let the south and west edges be connected into a path.

The boundary edges do not differ between fully packed loops, and so the maximum number of edges which differ at vertices incident to the boundary between  $\phi$  and  $\phi'$  is 2. It follows that such paths along which  $\phi$  and  $\phi'$  differ must close up to form loops with possible osculations

at the degree 4 vertices. Thus the set of edges of  $\phi$  and  $\phi'$  which differ in color is a union of alternating loops.  $\square$

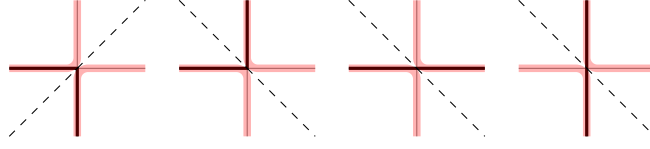


FIGURE 5. Possible osculations for alternating paths.

**Lemma 3.2.** *Any alternating loop can be constructed by some sequence of plaquette flips.*

*Proof.* The following proof works for any given fully packed loop. With a fixed fully packed loop and alternating path in mind, it is clear that if we can apply a single plaquette flip to all boxes in the interior, then each interior edge is flipped twice, while each exterior edge is only flipped once. Thus, such a sequence of plaquette flips implements an alternating path.

Let us call a box *accessible* if it can be flipped by a plaquette flip eventually, after some sequence of plaquette flips in the interior. We wish to demonstrate that all plaquette flips in the interior of an alternating path are accessible. The proof is by induction on the number of boxes in the interior. The case of one box is obvious, since this is simply a plaquette flip to begin with. The inductive step is demonstrated by cutting up the interior of the alternating path into two parts, where we cut along some alternating path. Then one of the two regions is bounded by an alternating path and the other region can be shown to be bounded by an alternating path upon a color flip operation applied to the cutting path. The number of boxes that each of these smaller alternating path bounds is smaller than  $n$ , therefore by the inductive hypothesis, all of the boxes within are accessible. Lastly, to demonstrate that such an alternating cutting path exists, we make the observation that every edge is part of some alternating path, by the six-vertex condition. Then, if we pick any edge that is both incident to a vertex on the alternating path and in the interior of the alternating path and use this edge to find a new alternating cutting path, we see that the cutting path must be incident to our original alternating path in at least 2 points.  $\square$

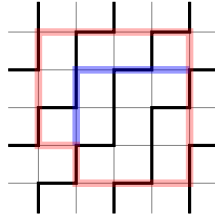
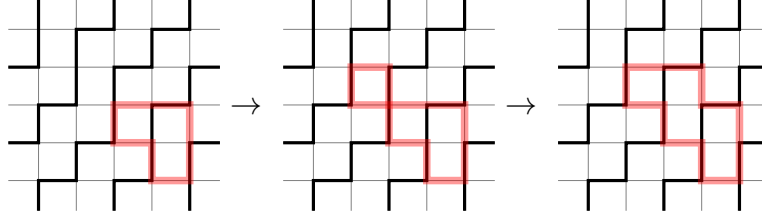


FIGURE 6. The path colored blue illustrates one possible cutting path.

Now in all that follows we shall fix  $\phi$  to be the fully packed loop associated to the identity matrix,  $\phi_0$ . We now show that the set of alternating paths on  $\phi_0$  have the structure of level set boundaries of Dyck islands.

**Theorem 3.3.** *Alternating paths on  $\phi_0$  are Dyck islands.*

*Proof.* We first observe that local update laws for alternating paths correspond to those for Dyck islands. This is perhaps best illustrated in the following sequence of plaquette flips:



Then by Lemma 3.2, any alternating loop can be implemented via a sequence of plaquette flips. Thus it will suffice to show that the level set boundaries of Dyck islands are in fact alternating paths. This follows from the structure of  $\phi_0$ , since every vertex away from the diagonal is of vertex type 3 or 4.  $\square$

#### 4. THE LATTICE STRUCTURE OF ALTERNATING PATHS ON $\phi_0$

We define a *lattice* to be a poset,  $L$ , with two binary operations  $\vee$  and  $\wedge$  such that  $x \vee y$  and  $x \wedge y$  are respectively the least upper bound and greatest lower bound of the elements  $x$  and  $y$ .

The partial ordering on Dyck islands is given by a box-wise comparison of the entries. If  $\gamma$  and  $\delta$  are Dyck islands, then we say that  $\gamma \leq \delta$  if and only if  $\gamma_{ij} \leq \delta_{ij}$  for all  $i$  and  $j$ . Using this notion of ordering, it is clear that the set of all Dyck islands for fully packed loops of size  $n$  forms a lattice, which we choose to denote in the sequel by  $\Delta_n$ . Furthermore, in analogy to Young's lattice, one can see that for every  $n$ , we have that  $\Delta_n$  is a distributive lattice.

Let  $\delta^{\phi_0}$  and  $\delta^{\phi_1}$  be the Dyck islands associated to  $\phi_0$  and  $\phi_1$ , respectively.

**Proposition 4.1.** *In the lattice of Dyck islands  $\Delta_n$ , the element  $\delta^{\phi_0}$  is  $\hat{0}$  and  $\delta^{\phi_1}$  is  $\hat{1}$ .*

*Proof.* It is clear that  $\delta^{\phi_0}$  is  $\hat{0}$ . In order to see that  $\delta^{\phi_1}$  is  $\hat{1}$ , we first demonstrate by induction that  $\delta^{\phi_1}$  is given as the union of  $\lfloor \frac{n}{2} \rfloor$  square loop alternating paths. Let us first consider the case when  $n = 2m - 1$  is odd and then induct on  $m$ . Supposing now that in the case of  $n = 2m - 3$ ,  $\delta^{\phi_1}$  consists of concentric square loop alternating paths, we simply need to demonstrate that by padding an extra row (or column) to each side in such a way that gives  $\phi_1$  for  $2m - 1$  gives an extra square loop alternating path on the exterior.

In the alternating sign matrix picture, we construct the correct matrix by padding the matrix on all sides with one row (column) of zeros except for two ones which are placed in the corners to give the skew-identity matrix. In the fully packed loop model, this is exactly the desired alternating path.

In order to show that this is maximal, we simply observe that for any box  $\delta_{ij}$ , the maximal height is given by  $\min(i, j, n - i, n - j)$ , since this is the maximal number of alternating paths between  $\delta_{ij}$  and the boundary. Furthermore, this maximal height is attained by  $\delta^{\phi_1}$  for all  $1 \leq i, j \leq n$ .  $\square$

Let  $\Delta'_n$  be the set of all alternating paths of  $\phi_1$ . As a fully packed loop,  $\phi_1$  is a copy of  $\phi_0$  reflected across the vertical axis (and an additional color flip to all edges if  $n$  is even), so  $\Delta'_n$  must be the set of all Dyck islands reflected across the vertical axis.

**Definition 4.2.** Let  $\iota_n$  be the map on  $(n - 1) \times (n - 1)$  square tableaux given by

$$[\iota_n(\delta)]_{ij} = \min(i, j, n - i, n - j) - \delta_{ij}$$

$$\delta^{\phi_0} = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \delta^{\phi_1} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 2 & 1 \\ \hline 1 & 2 & 2 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}$$

FIGURE 7.  $\delta^{\phi_0}$  and  $\delta^{\phi_1}$  in  $\Delta_5$ .

In words,  $\iota_n$  replaces the height from of  $\delta$  with the distance from the maximal height.

**Proposition 4.3.**  $\iota_n$  maps  $\Delta_n$  to  $\Delta'_n$  and  $\Delta'_n$  to  $\Delta_n$ .

*Proof.* We will observe that under the involution  $\iota_n$ , inside corners in  $\Delta_n$  get mapped to outside corners in  $\Delta'_n$  and vice versa. Let us first consider the case that  $i$  and  $j$  are chosen such that  $\delta_{ij}$  is off of the diagonal and skew diagonal and its neighbors are given as

$$\begin{array}{|c|c|c|} \hline & \ell-1 & \\ \hline \ell & \ell & \ell-1 \\ \hline & \ell & \\ \hline \end{array} \xleftrightarrow{\iota_n} \begin{array}{|c|c|c|} \hline & \alpha & \\ \hline \alpha & \alpha & \alpha+1 \\ \hline & \alpha+1 & \\ \hline \end{array}$$

where  $\alpha = [\iota_n(\delta)]_{ij} = \min(i, j, n-i, n-j) - \ell$ ,  $\ell = \delta_{ij}$ , and we have assumed that  $\delta_{ij}$  is an outside corner located above the diagonal and above the skew-diagonal. We see that it gets mapped to an inside corner in the corresponding element of  $\Delta'_n$ . The other cases of an inside corner and boxes in other regions are proved analogously.  $\square$

**Proposition 4.4.** The map  $\iota_n$  reverses ordering. That is, if  $\delta \leq \delta'$ , then  $\iota_n(\delta) \geq \iota_n(\delta')$ .

*Proof.* If  $\delta \leq \delta'$ , then there is some sequence of Dyck islands where at each step, we change an inside corner into an outside corner until we obtain  $\delta'$ . Applying  $\iota_n$  to this sequence gives a sequence going from  $\iota_n(\delta)$  to  $\iota_n(\delta')$  by changing an outside corner into an inside corner at each step. The result follows.  $\square$

The properties of the involution  $\iota_n$  then result in the following corollary.

**Corollary 4.5.**  $\Delta_n$  is a self-dual distributive lattice.

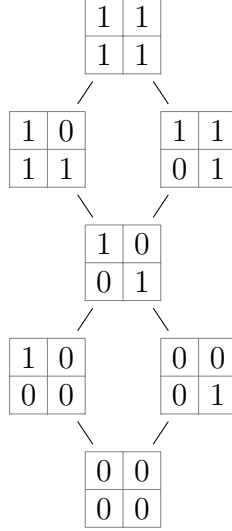
*Proof.* Let  $\tau$  be the map on square tableaux implementing a flip across a vertical axis. Then clearly we have that  $\tau$  maps  $\Delta_n$  to  $\Delta'_n$  and  $\Delta'_n$  to  $\Delta_n$  and so the composition  $\tau \circ \iota_n$  is an antiautomorphism of  $\Delta_n$ .  $\square$

## 5. ALTERNATING PATHS AND INVERSION NUMBER

**Definition 5.1.** The *inversion number* of an alternating sign matrix  $A$ , denoted by  $\text{inv}(A)$ , is

$$\text{inv}(A) = \sum_{\substack{1 \leq i, i', j, j' \leq n \\ i > i' \\ j < j'}} A_{ij} A_{i'j'}$$

The inversion number is an extension of the standard notion for permutation matrices to all alternating sign matrices. We shall often abuse notation and speak of the inversion number of the associated fully packed loop or Dyck island.

FIGURE 8. The lattice  $\Delta_3$ .

By only considering non-zero terms, the above sum can be reduced to a sum over pairs of integer tuples  $(i, j)$  and  $(i', j')$  such that  $(i', j')$  is strictly above and strictly to the right of  $(i, j)$  and such that  $A_{ij}$  and  $A_{i'j'}$  are non-zero. It is easy to see that under a flip across the diagonal, such pairs and the value of  $A_{ij}A_{i'j'}$  is preserved. Thus, the following lemma is true.

**Lemma 5.2.** *Let  $A'$  be the reflection of  $A$  across the diagonal. Then  $\text{inv}(A) = \text{inv}(A')$ .*

We now give a way to characterize the inversion number in terms of the loops of a Dyck island. Let  $\gamma$  be a loop of a given Dyck island  $\delta$ . We shall abuse notation in writing  $\text{inv}(\gamma)$  to mean the inversion number of a new Dyck island consisting of the single loop,  $\gamma$ .

**Theorem 5.3.** *If  $\delta$  is a Dyck island consisting of  $\ell$  loops (possibly nested),  $\gamma_1, \dots, \gamma_\ell$  with a total number of  $k$  off-diagonal osculations, then*

$$\text{inv}(\delta) = \sum_{n=1}^{\ell} \text{inv}(\gamma_n) - k$$

Before proving the theorem, we prove a lemma about evaluating a Dyck island consisting of only one loop  $\gamma$ .

Recall that the *semilength* of a Dyck path of  $2n$  steps ( $n$  rises and  $n$  falls) is  $n$ . If the northeast and southwest boundaries of a loop,  $\gamma$ , are Dyck paths of semilength  $n$ , then we say also that the semilength of  $\gamma$  is  $n$ . Furthermore, we define an *internal one* to be a diagonal point of the lattice which is strictly contained inside the interior of the loop. In the alternating sign matrix picture, these points correspond to entries along the diagonal with the value 1.

The analysis of inversion numbers of Dyck islands will be facilitated by the Dyck path structure of the northeast and southwest boundaries of loops. The next lemma characterizes a key property of subpaths of Dyck paths encoded as words in a two letter alphabet.



**Lemma 5.4.** *Let  $w$  be a word,  $w$ , in the alphabet  $\{u, d\}$  such that  $w$  begins with the letter  $u$  and ends with the letter  $d$ . Then the number of  $ud$  subwords minus the number of  $du$  subwords is exactly 1.*

*Proof.* Start with the word  $ud$ , which obviously evaluates to 1. The insertion of a single  $u$  or  $d$  in the middle of the word does not change this evaluation. Thus, by checking that the following four insertion scenarios do not change the evaluation, we are done.

$$\begin{aligned} uu &\leftrightarrow uuu \\ ud &\leftrightarrow uud \\ du &\leftrightarrow duu \\ dd &\leftrightarrow dud \end{aligned}$$

The case for insertion of a  $d$  is exactly analogous. □

**Lemma 5.5.** *Suppose that  $\gamma$  has  $m$  internal ones. Then*

$$\text{inv}(\gamma) = \text{semilength}(\gamma) + m$$

*Proof.* We first make the observation that outside corners correspond to the value 1 in the alternating sign matrix picture and that inside corners correspond to the value  $-1$ . The non-zero terms in the definition of the inversion number then correspond to a sum over pairs of corners, with one corner taken from the southwest boundary and paired with one corner from the northeast boundary. Thus, we see immediately that the inside corners along the diagonal where the northeast boundary and southwest boundary touch do not contribute to the sum.

The proof then proceeds by demonstrating that the formula holds for a fixed southwest boundary and arbitrary northeast boundary which preserves the number of internal ones and semilength. Then one can see that the formula holds for an arbitrary southwest boundary that fixes the number of internal ones and semilength as well, by the invariance of the inversion number under diagonal flips given by Lemma 5.2.

Now pick a corner along the southwest boundary. Let us denote this corner by  $x$ . Isolate the relevant part of the northeast boundary which is above and to the right of  $x$ . This path will correspond to a word (possibly the empty word) in the alphabet  $\{u, d\}$  such that  $u$  corresponds rises and  $d$  corresponds to falls. Then one sees by an analysis on words corresponding to relevant northeast boundaries that the contribution to the inversion sum is equal to  $\text{height}(x)$  for outside corners, and  $-\text{height}(x)$  for inside corners. In similar fashion, inside ones each contribute a value of 1 to the sum, by Lemma 5.4. Finally, to finish the proof, it suffices to show that the contribution to the inversion number from the southwest boundary is

$$\sum_{x, \text{ outside corners}} \text{height}(x) - \sum_{y, \text{ inside corners}} \text{height}(y) = \text{semilength}(\gamma)$$

This is certainly true in the event that there is only one corner (it must be an outside corner) at height  $n$ . This corresponds to the Dyck word  $uu \dots udd \dots d$ , where there is an outside corner at all instances of the subword ‘ $ud$ ’ and an inside corner at all instances of the subword ‘ $du$ ’. The height of this corner is given by  $\#u - \#d \pm 1$ , where  $+$  is for outside corners and  $-$  is for inside corners and  $\#u$  and  $\#d$  count the instances of the letters  $u$  and  $d$  before

the subword in consideration. Thus for our special case of one outside corner, the equation holds. Then it is easy to check that the sum above is invariant under the local changes  $ud \leftrightarrow du$ .  $\square$

*Proof of Theorem 5.3.* For loops which are disjoint and not nested, the formula is clear. To deal with loops which are nested, but have no off-diagonal osculations, we claim that we can evaluate the southwest boundaries (swb) and northeast boundaries (neb) of loops according to the formulas:

$$\begin{aligned}\text{swb}(\gamma) &= \text{semilength}(\gamma) + N(\gamma)(1 + \#\{\mathcal{O}_{sw}^\gamma\}) \\ \text{neb}(\gamma) &= N(\gamma)(1 + \#\{\mathcal{O}^\gamma \setminus \mathcal{O}_{sw}^\gamma\})\end{aligned}$$

where  $N(\gamma)$  is the number of loops which contain  $\gamma$  in its interior, neglecting points of osculation, and  $\mathcal{O}^\gamma$  is the set of all vertices along the diagonal which touch  $\gamma$  possibly along both the southwest and northeast boundaries, and  $\mathcal{O}_{sw}^\gamma$  is the set of all vertices along the diagonal which touch the southwest boundary of  $\gamma$ . A justification of these formulas will be provided in Lemma 5.6.

Then if  $\delta$  is the Dyck island consisting of the loops  $\gamma_1, \dots, \gamma_\ell$  with inside ones labeled  $p_1, \dots, p_m$  and with no off-diagonal osculations, we have the following evaluation of  $\text{inv}(\delta)$ :

$$\begin{aligned}\text{inv}(\delta) &= \sum_{i=1}^{\ell} \text{swb}(\gamma_i) + \text{neb}(\gamma_i) + \sum_{j=1}^m N(p_j) \\ &= \sum_{i=1}^{\ell} \text{semilength}(\gamma_i) + N(\gamma_i)(2 + \#\{\mathcal{O}^{\gamma_i}\}) + \sum_{j=1}^m N(p_j) \\ &= \sum_{i=1}^{\ell} \text{inv}(\gamma_i)\end{aligned}$$

where  $N(p_j)$  is the number of loops which contain the vertex  $p_j$ . The last equality holds because the number of diagonal osculations exactly account for the inside ones that would have been there otherwise. Also, the value 2 accounts for the two inside ones which would have been in place of the left and right endpoints of the northeast and southwest boundaries of  $\gamma_i$ .

Lastly, in order to account for off-diagonal osculations, for each osculation, the inside corner would normally contribute some value  $-i$  to the sum, and the outside corner would normally contribute  $i + 1$ , since it is nested inside one additional loop and is at the same height. Then since the point of osculation is a 0 in the alternating sign matrix picture, these two numbers are not contributed to the sum. Furthermore, since the osculation cancels out a pair of alternating sign matrix entries 1 and  $-1$  and merges them into a 0 entry, we see that all contributions from inside ones, inside corners and outside corners to the southwest are unaffected. Thus in the end, the sum  $\sum_{i=1}^{\ell} \text{inv}(\gamma_i)$  overestimates the inversion number by exactly the number of off-diagonal osculations,  $k$ . Thus the equation holds.  $\square$

**Lemma 5.6.**

$$\begin{aligned}\text{swb}(\gamma) &= \text{semilength}(\gamma) + N(\gamma)(1 + \#\{\mathcal{O}_{sw}^\gamma\}) \\ \text{neb}(\gamma) &= N(\gamma)(1 + \#\{\mathcal{O}^\gamma \setminus \mathcal{O}_{sw}^\gamma\})\end{aligned}$$

*Proof.* Suppose that  $\gamma$  is in the interior of  $N$  loops. First, let us prove the northeast boundary equation. Suppose that  $w$  is a Dyck word representing the northeast boundary of  $\gamma$ . Each ‘ $ud$ ’ contributes  $+N$  and each ‘ $du$ ’ contributes  $-N$ . Thus by Lemma 5.4, the total contribution is  $N$ . Additionally, when there is a diagonal osculation which is not in  $\mathcal{O}_{sw}^\gamma$ , then the corresponding ‘ $du$ ’ does not contribute a  $-N$  since there is a 0 in place of a 1 in the alternating sign matrix picture. This establishes the northeast boundary.

For the southwest boundary, let  $w$  be the Dyck word to which it corresponds. Each instance of ‘ $ud$ ’ contributes  $(N + \text{height})$  and each instance of ‘ $du$ ’ contributes  $(-N - \text{height})$ . So this word evaluates to  $\text{semilength}(\gamma) + N$ . Observe that all of the double diagonal osculations which correspond to  $-1$  in the alternating sign matrix picture have been accounted for in the northeast boundary. Hence, every diagonal osculation will contribute 0 instead of  $-N$ . Thus the southeast boundary equation is established.  $\square$

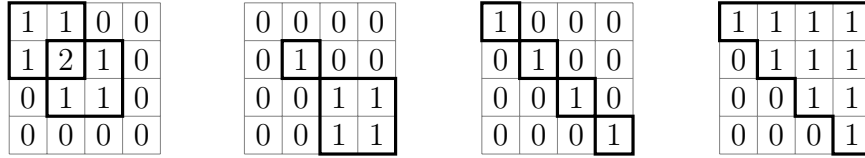


FIGURE 9. Some example Dyck islands with inversion number 4

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